

Homework 1

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Question 1:

This question concerns a game between two players: Alice who has a finite strategy set S and Bob who has a finite strategy set T . We will denote Alice's utility when she plays s and Bob plays t by $a(s, t)$ and Bob's utility by $b(s, t)$. We start with the following definitions of strict dominance:

- s is *purely-dominated* if for some $s' \in S$ we have that for all $t \in T$: $a(s, t) < a(s', t)$.
- s is *mixed-dominated* if for some $x \in \Delta(S)$ we have that for all $t \in T$: $a(s, t) < a(x, t)$.
- s is *never-best-reply* if for every $t \in T$ there exists $s' \in S$ so that: $a(s, t) < a(s', t)$.
- Similar, dual, notions apply for the strategies of Bob.

1. Prove: s is purely-dominated $\Rightarrow s$ is mixed-dominated $\Rightarrow s$ is never-best-reply.
2. Give examples showing that the opposite implications are false.
3. Show that all three notions can be computed in polynomial time.
4. We could have also defined s to be a *never-best-reply-to-mixed* if for every $y \in \Delta(T)$ there exists $s' \in S$ so that: $a(s, y) < a(s', y)$. Prove that this is equivalent to s being mixed-dominated. (Hint: define a zero-sum game using Alice's utility.)

Question 2:

1. Show that the following problem can be solved in polynomial time: Given a two-player game, a subset of the rows S and a subset of the columns T , find a mixed-Nash equilibrium where the support of the row strategy is exactly S and the support of the column strategy exactly T , or state that such an equilibrium does not exist.
2. Show that there exists an exponential time algorithm for computing a mixed-Nash equilibrium in two-player games.

Question 3:

This question deals with the following bandwidth selection game: There are n radio users, where each user i needs to transmit q_i bits of information. There are two available radio bands, and each radio user can choose which band to use. The radios that use a certain band b share the bandwidth of the band, and all of them finish together at time. The two bands are equivalent for all purposes and the only thing that the users want is to finish as early as possible.

1. Formalize this as a game.
2. Show that the allocation of users to bands that minimizes the makespan (the load on the most loaded band) is a pure Nash equilibrium.
3. Give an example where there is a pure equilibrium that does not minimize the makespan.
4. Prove that if we start from an arbitrary partition of the users to the two bands and then repeat the following step sufficiently many times (but only finitely many times), we reach a pure Nash equilibrium. Step: take an arbitrary user that can improve his utility by moving to the other band and move him there. reaches equilibrium.
5. Show that finding the allocation that minimizes the makespan is NP-complete (hint: reduction from partition.)
6. Give a polynomial time algorithm to find a pure Nash equilibrium.
7. Assume now that, as opposed to all previous parts of the question, each user may partition his radio transmission between the two bands (e.g. sending $q_i/3$ in band 1 and $2q_i/3$ in band 2). Find strategies for the users that (1) are in equilibrium (2) together minimize the makespan (3) each user can decide what to do without looking at the others loads.

Question 4:

Consider n machines and m selfish jobs (the players). Each job j has a processing time p_j and a set S_j of machines on which it can be scheduled (i.e., S_j is the strategy space of player j). Once all jobs have chosen machines, the jobs on each machine are processed *serially* from shortest to longest. (you can assume that the p_j 's are distinct). For example, if jobs with processing times 1,3, and 5 are schedules on a common machine, then they will complete at times 1, 4, and 9, respectively. Assume that players choose machines in order to minimize their completion times.

Consider the following scheduling algorithm: (1) Sort all the jobs in order from smallest to largest; (2) Schedule the jobs one-at-a-time, assigning a job j to the machine of S_j with minimum load so far (breaking ties arbitrarily). Prove that the pure Nash equilibria of the scheduling game are precisely the possible outputs of this scheduling algorithm (with the different equilibria arising from different ways of breaking ties). [hint: if you were the smallest player, how is your personal cost affected by the others' decisions?].