| **Algorithmic Game Theory** | Due: May 18th, 2016 |
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# Homework 2

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## 1 Auctions

**Problem 1:** Consider a second price auction with $n$ bidders and suppose a subset $S$ of the bidders decide to collude, meaning that they submit false bids in a coordinated way to maximize the sum of their payoffs. Establish necessary and sufficient conditions on the set $S$ (in terms of the private valuations of the bidders) such that the bidders of $S$ can increase their collective payoff via non-truthful bidding.

**Problem 2:** Consider a combinatorial auction with a set $M$ of $m$ goods and $n$ bidders. Assume that the valuation function of every bidder $v_i(\cdot)$ is normalized, monotone, and *subadditive* (i.e., for every disjoint sets $T_1, T_2$, $v_i(T_1) + v_i(T_2) \geq v_i(T_1 \cup T_2)$).

Consider the winner determination problem, and for now, ignore payments and truthfulness, rather consider only poly-time social welfare maximization. Given $M$ and $v_1, \ldots, v_n$, call the winner determination problem *lopsided* if there is an optimal allocation of goods in which at least half of the total SW of the allocation is due to players that were allocated a bundle with at least $\sqrt{m}$ goods. (i.e., if $\sum_{i \in A} v_i(T_i^*) \geq \frac{1}{2} \sum_{i=1}^{n} v_i(T_i^*)$, where $T^*$ is the optimal allocation and $A$ is the subset of bidders $i$ with $|T_i^*| \geq \sqrt{m}$.)

1. Show that in a lopsided problem, there is an allocation that gives all the goods to a single player and achieves an $\Omega(1/\sqrt{m})$ fraction of the maximum-possible SW.
2. Show that in a problem that is not lopsided, there is an allocation that gives at most one good to each player and achieves an $\Omega(1/\sqrt{m})$ fraction of the maximum-possible SW. [hint: use subadditivity.]
3. Give a poly-time $O(\sqrt{m})$-approximate winner determination algorithm for subadditive valuations. [hint: make use of a graph matching algorithm].
4. Give a poly-time $O(\sqrt{m})$-approximate, truthful combinatorial auction for subadditive valuations.

## 2 Sponsored Search

**Problem 3:** Consider the following extension of the sponsored search setting. Each bidder i now has a publicly known *quality* $\beta_i$ (in addition to a private valuation $v_i$ per click). As usual, each slot $j$ has a click-through-rate (CTR) $\alpha_j$, and $\alpha_1 \geq \alpha_2 \ldots \geq \alpha_k$. We assume that if bidder $i$ is placed in slot $j$, its probability of a click is $\beta_i \alpha_j$ – thus, bidder $i$ derives value $v_i \beta_i \alpha_j$ from this outcome.

Describe the surplus-maximizing allocation rule in this generalized sponsored search setting. Argue that this rule is monotone. Give an explicit formula for the per-click payment of each bidder that extends this allocation rule to a DSIC mechanism.

## 3 VCG

**Problem 4:** Consider a combinatorial auction in which a bidder can submit multiple bids under different names, unbeknownst to the mechanism. The allocation and payment of a bidder is the union and sum of the allocations and payments, respectively assigned to all of its pseudonyms. Show that the following is possible: a bidder in a combinatorial auction can earn higher utility from the VCG mechanism by submitting multiple bids than by bidding truthfully. Can this ever happen in the Vickrey auction? Give a brief explanation.

**Problem 5:**

(a) A single seller sells $k$ identical goods. Each one of $n$ bidders is willing to buy a single good, and has a private valuation $v_i$ for the good (or for any bundle including that one good).

   i. Describe the allocation and the payments of the VCG mechanism with Clarcke Pivot rule.

   ii. Design an algorithm that finds the optimal allocation (without enumerating over all possibilities).

(b) Prove that no bidder can increase his utility in the VCG mechanism by submitting another false bid under different name (as in question 4).

## 4 Smooth Games

**Problem 6:** Prove that if s is an $\epsilon$-approximate Nash equilibrium of a $(\lambda, \mu)$-smooth cost-minimization game – meaning that $C_i(s) \leq (1+\epsilon)C_i(s_i', s_{-i})$ for every player $i$ and deviation $s_i' \in A_i$ – with $\epsilon < \frac{1}{\mu} - 1$ then the cost of $s$ is at most $\frac{\lambda(1+\epsilon)}{1-\mu(1+\epsilon)}$ times that of an optimal outcome.