Algorithmic Game Theory	June 8, 2016
Lecture 13	
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1 The Prophet Inequality

1.1 Reminder:

In the last class, we define a game in which we have n levels. In the $i'th$ level we suggest a prize $_i$ to the player, and he needs to decide if he wishes to stop the game, and to get this prize, or to stay for the next level. If the player chooses to stay for the next levels, he loses his right to regret later, and to ask this prize (if the next prizes will be smaller than this prize).

Theorem 1 (The Prophet Inequality) *Every independent distributions $G_1, , G_n$ there exists a "simple" strategy t, gives in expectation at least $\frac{1}{2}E[max_i]$. Strictly speaking, this simple strategy is a threshold strategy t. The strategy tells us in the $i'th$ level, to get the prize iff $_it$*

Definition 2 *denote $z^+ = max[0, z]$*

Definition 3 *denote $q(t) = Pr(obtaining\ no\ prize)$*

Proof: We are going to show the following:

1. We shall express a lower bound on the utility of the player.
2. We shall express an upper bound on the utility of the benchmark.
3. We shall connect them, for getting the desired bound.

Proofs:
1.

$$
\begin{aligned}
&E[\text{utility from threshold strategy t}] \\
&\geq (1-q(t))t + \sum_{i=1}^{n} E[_i - t_i t, ji, _i < t] Pr(_i t) Pr(_j < tji) \\
&= (1-q(t))t + \sum_{i=1}^{n} E[_i - t_i t] Pr(_i t) Pr(_j < tji) \\
&\geq (1-q(t))t + q(t) \sum E[(_i - t)^+]
\end{aligned}
$$

2.

$$E[\max{(_i)}] = E[t + \max{(_i - t)}]t + E[\max{(_i - t)^+}]t + \sum E(_i - t)^+ \tag{1}$$

3. The desired bound is obtained taking t satisfies $q(t) = 1/2$. ■

Example:

$$
\begin{aligned}
&\text{If } \pi_1, \pi_2 \sim U[0,1], \text{ then } t = \frac{1}{\sqrt{2}}, \\
&\text{since: } q(\frac{1}{\sqrt{2}}) = Pr(_1, _2 \frac{1}{\sqrt{2}}) \\
&= Pr(_1 \frac{1}{\sqrt{2}}) Pr(_2 \frac{1}{\sqrt{2}}) \\
&= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \\
&= \frac{1}{2}
\end{aligned}
$$

2 Implications:

2.1 One item auction:

$E[Rev] = E[\text{virtual welfare}]$ Where the virtual utility is defined by:$(v) = 1 - (1 - F(v))/f(v)$. As the notations learned in the last classes. Let us connect between the one item auction with players from distributions $F_1, , F_n$ by connecting $_i(v_i)^+$ to the i'th prize in the PI (=Prophet Inequality) problem.

In the optimal auction, it holds that: $E_v[_i(v_i)x_i(v)] = E_v[max_i(v_i)]$ Where the rhs is the expected prize of the prophet with prizes $_1(v_1)^+, , _n(v_n)^+$. We shall look at the following allocation:

- We shall choose t such that $Pr[max_i(v_i)^+t] = 1/2$.
- We shall give the item to the i'th player for which $_i(v_i)t$, if such a player exists.

Using this allocation, by the theorem from the last class, we know that virtual welfare achieve at least half the revenue. Formally, $E_v[_i(v_i)^+]1/2E_v[max_i(v_i)^+]$(since the rhs is the optimal revenue).

Example 1 *Let us define a minimal price r_i for every player i: $r_i = {}_i^{-1}(t)$ (for the above mentioned t). We shall give the product to the player whose price value exceeds the minimal price. Prior Independent Auctions*

3 Prior dependent auctions

VCG is referred to as "$prior - free$", since $\{v_i\}_{i=1}^n$ are constant, rather than random variables taken from some distribution. The algorithm we saw for optimal auction which maximizes revenue, is prior dependent, since $\{v_i\}_{i=1}^n$ are taken from some known distribution.
Now, we introduce a third kind of auctions, that are so-called "prior independent auctions". In this kind of auctions, the values $\{v_i\}_{i=1}^n$ are taken from unknown (to the seller) distributions $F_1, , F_n$. This means that the seller cannot use the distributions for the auction planning. He only can use it to performance analysis.

Theorem 4 ***Theorem 2: [Bulow-Klemperer theorem]*** *Let F be regular distribution (unknown to the seller). It holds that the expected value of the revenue of the auction given by VCG with $n+1$ players, is greater than or equal to the expected value of the revenue of the optimal auction (that maximizes the revenue) with n players, formally:*

$$E_{v_1,,v_{n+1}\sim F}[Rev_{n+1}(VCG)] \geq E_{v_1,,v_{n+1}\sim F}[Rev_n(OPT_F)]$$

Proof:
Auction definition: Let us look at an auction M with $n+1$ players: M runs OPT_F on the first n players. If the product wasn't given to any of the first players we give it for free to the $(n+1)$th player. Clearly, the auction M always allocates the product to one of the players.

Auction properties: It holds that:

$$(*)E[Rev_n(M)] = E[Rev_n(OPT_F)]$$

By the definition of OPT_F which is a second price auction, with minimal price $\phi^{-1}(0)$, according to Meyerson theorem.

Additionally, it holds that: For every auction M' that always allocates the product to some player, it holds that:

$$E[Rev_(n+1)(VCG)]E[Rev_n(M')]$$

In particular, taking $M' = M$, we get:

$$(**)E[Rev_(n+1)(VCG)]E[Rev_n(M)]$$

From $(*)$ and $(**)$ together, we obtain:

$$E[Rev_(n+1)(VCG)] \geq E[Rev_n(OPT_F)]$$

as desired. ■