Algorithmic Game Theory	March 9, 2016

Lecture 2

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1 Min-max Theorem

Reminder: Last time we've seen the theorem which states that every finite game has a mixed nash equilibrium. We've concentrated on two- player zero sum games, and stated the following lemma:

Lemma 1 $\max_i \min_j a_{i,j} \leq \min_j \max_i a_{i,j}$
(this essentially states that each player is better off playing second).

Proof: It is sufficient to prove that for every i_0, j_0 the following holds:
$\min_j a_{i_0,j} \leq \max_i a_{i,j_0}$ (since then it also implies for i_0 set to the argmax of left side of equation and j_0 set to the argmin of the right side of the equation, so the equation follows)
Indeed, for each i_0, j_0 $\min_j a_{i_0,j} \leq a_{i_0,j_0} \leq \max_i a_{i,j_0}$. ■

Notation:
$x = (x_1, ..., x_n)$ will note the mixed strategy of the row player.
$y = (y_1, ..., y_n)$ will note the mixed strategy of the column player.

Observation 2 *Given a pair of mixed strategies (x, y), the value of the row player will be:*

$$\sum_i \sum_j x_i y_j a_{i,j}$$

which is the expectation of the utility of the rows player.

Observation 3 *Given a strategy x of player 1, player 2 will W.L.O.G choose a pure strategy j (since each y_i in the support of y has the same expected value) that minimizes:*

$$\sum_i x_i a_{i,j}.$$

And therefore player 1's best strategy is choosing:

$$x = arg \max_s \min_j \sum_i s_i a_{i,j}.$$

Similarly, given a strategy y of player 2, player 1 will W.L.O.G choose a pure strategy i that maximizes:

$$\sum_j y_j a_{i,j}.$$

And therefore player 2's best strategy is choosing:

$$y = arg \min_r \max_i \sum_j r_j a_{i,j}.$$

Theorem 4 (min-max) *Let $x = \{x_1, ..., x_n\}$ and $y = \{y_1, ..., y_m\}$ be mixed strategies. Then*

$$\max_x \min_j \sum_i x_i a_{i,j} = \min_y \max_i \sum_j y_j a_{i,j}.$$

Proof: The optimization problem that the row player (player 1) tries to solve is as follows:
Problem (1):

$$\max_x \min_j \sum_i x_i a_{i,j}$$

such that

$$\sum_i x_i = 1$$

$$\forall i : x_i \geq 0.$$

Observe that problem (1) is equivalent to the following LP problem:
Problem (2):

$$\max c$$

such that

$$\forall j : \sum_i x_i a_{i,j} \geq c$$

$$\sum_i x_i = 1$$

$$\forall i : x_i \geq 0.$$

The optimization problem that the column player (player 2) tries to solve is as follows:
Problem (1'):

$$\min_y \max_i \sum_j y_j a_{i,j}$$

such that

$$\sum_i y_i = 1$$

$$\forall j : y_j \geq 0$$

Problem (1') is equivalent to the following LP problem:
Problem (2'):

$$\min d$$

such that

$$\forall i : \sum_j y_j a_{i,j} \leq d$$

$$\sum_j y_j = 1$$

$$\forall j : y_j \geq 0.$$

We state the following two LP problems:
LP (3):

$$\min_x \sum_i x_i$$

such that

$$\forall j \sum_i x_i a_{i,j} \geq 1$$

$$\forall i : x_i \geq 0.$$

LP (3'):

$$\max_y \sum_j y_j$$

such that

$$\forall i \sum_j y_j a_{i,j} \leq 1$$

$$\forall j : y_j \geq 0.$$

Notation : We note the optimal value of the optimization problem (x) by $opt_{(x)}$.

Claim 5

$$OPT_{(3)} = 1/OPT_{(2)}$$

Proof: $\leq$: Let x be a feasible solution for (2) with the value c. Therefore, $x/c = (x_1/c, ..., x_n/c)$ is a feasible solution for (3) (inferred from the first equation of (2)), with value $1/c$ (inferred from the second equation of (2)).
$\geq$: Let x be a feasible solution for (3) with the value $1/c$. Therefore, $x * c = (x_1 * c, ..., x_n * c)$ is a feasible solution for (2) (inferred from the first equation of (3), and the value of (3)), with value $1/c$. ■

Claim 6

$$OPT_{(3')} = 1/OPT_{(2')}$$

Proof: Similarly to previous claim. ■

To finish the proof, we note that (3),(3') are dual LP problems, and from the duality theorem we conclude that they have the same optimal value. therefore:

$$1/OPT_{(1)} = 1/OPT_{(2)} = OPT_{(3)} = OPT_{(3')} = 1/OPT_{(2')} = 1/OPT_{(1')}.$$

($OPT_{(1)} = OPT_{(2)}$ because they represent the same LP problem. Similarly, $OPT_{(1')} = OPT_{(2')}$)
We conclude that $OPT_{(1)} = OPT_{(1')}$. ■

The meaning of the min-max theorem is that for every two- player zero- sum game there exists a value v, defined by the value achieved in the equality of the theorem. The row player can achieve v as a lower bound if he plays first, and the column player can achieve v as an upper bound (for the row player) if he plays first.

Corollary 7 *The strategy profile (x, y) which achieves the optimum is a NE.*

Proof: Assuming player 1 plays x, player 2 cannot give a better response than y (where he achieves score v), since player 1 has v as a lower bound by playing x. Assuming player 2 plays y, player 1 cannot give a better response than x (where he achieves score v), since player 2 has v as an upper bound by playing y. ■

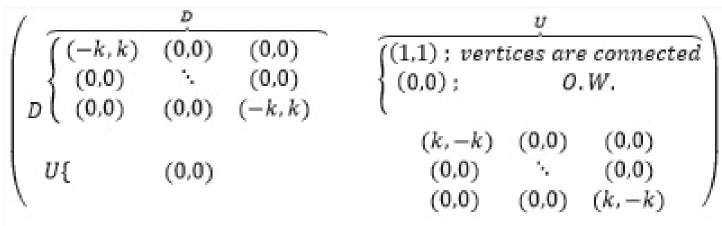
Corollary 8 *It is possible to compute a NE in polynomial time for every Zero-Sum game. (since computing such a NE is equivalent to solving the LP problems (i.e. opt_2)above, and an LP problem is solvable in polynomial time with the ellipsoid method)*

2 NP-Hardness of finding a NE

Fact 9 *The following problem is NP-Hard:*
Input: a bipartite graph G with sides U, D and a number k.
Problem: Are there two subsets of vertices $U' \subset U, D' \subset D$ both of size k such that they form a bi-clique ($\{a, b\} \in E$ for all $a \in U'$ and $b \in D'$).

Theorem 10 *The following problem is NP-HARD:*
In a 2-player game, where player 1 and player 2 both have m strategies, tell whether there is a NE where the sum of the utilities is ≥ 2.

Proof:
We show a reduction from the bipartite bi-clique problem to the given problem.
Given a bipartite graph $G = ((U, D), E)$ where $D = (d_1, .., d_n)$, $U = (u_1, .., u_m)$ and a number k, we build the following game:
Each player chooses a node from U or D.
If player 1 chooses $u \in U$ and player 2 chooses $d \in D$ then the utility is (1,1) if $(u, d) \in E$ otherwise it is (0,0).
If player 1 chooses $d \in D$ and player 2 chooses $u \in U$ then the utility is (0,0).
If player 1 chooses $d_1 \in D$ and player 2 chooses $d_2 \in D$ then the utility is (k,-k) if $d_1 = d_2$ and (0,0) otherwise.
Finally, if player 1 chooses $u_1 \in U$ and player 2 chooses $u_2 \in U$ then the utility is (-k,k) if $u_1 = u_2$ and (0,0) otherwise.
The following illustrates the games' matrix:



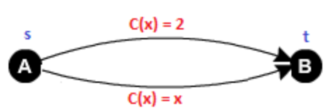
We now prove that G contains a (k,k) bi-clique $\iff$ the game has a NE with sum of utilities ≥ 2.
$\Longrightarrow$:
Assume there is a (k,k) bi-clique $U' \subset U$, $D' \subset D$. We focus on the following strategy profile:
Player 1 chooses uniformly from the vertices of U', and Player 2 chooses uniformly from the vertices of D'. We show the sum of utilities is ≥ 2 and that the strategy profile is a NE.
Each player has an expected value of 1:
Each result of such game has an outcome of (1,1) since there are edges from every node in U' to every node in D'. Therefore the expected value is 1 for both players.
The strategy profile is a NE:
Set player 2's strategy. We show player 1 cannot increase its utility by changing strategies. As we've seen we can assume W.L.O.G, that player 1 switches to a pure strategy.
Player 1's other possible responses are either (0,0), (1,1) or (k,-k). (k,-k) is clearly his best option, on the other hand player 2 picks that cell with probability $\frac{1}{k}$ meaning the utility is 1. Therefore player 1 can not increase its utility by changing strategies. Symmetrically player 2 can not increase its utility, meaning the strategy profile is a NE.
$\Longleftarrow$:
Assume we're given a strategy profile with sum of utilities ≥ 2 and that it is a NE. We show there is a (k,k) bi-clique.
Since the sum of utilities is ≥ 2, only values in the upper left corner can have positive probabilities. Since the sum of utilities is ≥ 2, and each outcome could contribute at most

2 to the value (in (1,1)), we conclude that each node in the support of player 1 is connected to each node in the support of player 2.Also, the amount of positive probabilities for each player is $\geq k$ (otherwise we have a player, assume player 1, with a strategy with probability $> \frac{1}{k}$ meaning player 2 could deviate to (-k,k) increasing his utility in contradiction to the fact that we're in a NE). Therefore, the nodes corresponding to the players' strategies form a bi-clique with at least k nodes at each side. ■

3 Atomic Selfish Routing

Definition 11 *An atomic selfish routing game between k players is defined as follows:*
- *G=(V,E) - a digraph.*
- *Every player, $i \in [k]$, has a source $s_i \in V$ and a target $t_i \in V$.*
- *Every player has one unit of traffic to pass from s_i to t_i.*
- *Every players' set of strategies is the set of paths from s_i to t_i in G.*
- *Every edge has a cost function $c_e(x_e) \geq 0$, x_e being the number of players using edge e.*

Example 1 *For k=2,*



The goal function is the sum of the cost of all the players.
- *The strategy profile where one player uses the upper path and another player uses the lower path is an equilibrium with cost 3, and this is the optimal strategy.*
- *The strategy profile where both players use the lower path is also an equilibrium. Its cost is 4 and therefore it is not optimal.*

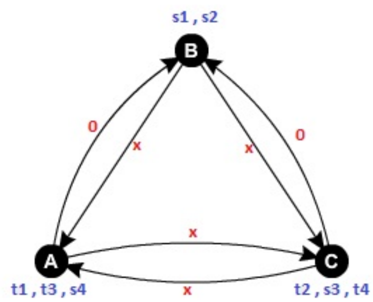
Conclusion: *In this game there are two different strategy profiles that are equilibrium. One of them has a cost of 3 (and it is optimal), the other has a cost of 4.*

Definition 12 *Given a game G, and a family of games $\mathcal{G}$ define the price of anarchy of G and $\mathcal{G}$ as follows:*
- *$POA(G) = \frac{cost\,of\,worst\,equilibriam(G)}{cost\,of\,optimal\,outcome(G)}$*
- *$POA(\mathcal{G}) = \sup_{G \in \mathcal{G}} POA(G)$*

In the example above, POA(G) $= \frac{4}{3}$.

Theorem 13 *Given an atomic, selfish, routing game G, with edges with affine costs (i.e. $\forall e \in E \;\; c_e(x_e) = a_e x_e + b_e,\; a_e, b_e \geq 0$) $\Rightarrow$ POA(G) $\leq \frac{5}{2}$.*

Observation 14 *The theorem is tight - the following game G has POA(G) $= \frac{5}{2}$:*



Clearly the OPT is reached when every player chooses its path to be one x edge $\Rightarrow$ cost(OPT) = 4.
On the other hand, let's look at the following strategy profile: each players' path is the only path that goes through 2 edges. For example, the path for s_1 would be B,C,A. This profile is clearly a NE and its cost is 10.
$\Rightarrow$ *POA(G)* $= \frac{10}{4} = \frac{5}{2}$.

Proof: (Of Theorem 13)
[Note we now focus strictly on **pure** strategies]
Let f be a flow (which essentially matches a profile strategy) in equilibrium, and let f* be an optimal flow (which essentially matches an optimal profile strategy).
For each $e \in E$,
• Let f_e be the number of players who chose the edge e in the flow f.
• Let f_e* be the number of players who chose the edge e in the flow f*.

Stage 1:
Let p_i be the route chosen by player i in f, and let p_i* be the route chosen by player i in f*.
The flow f is in equilibrium, so:

$$\forall i, \sum_{e \in p_i} c_e(f_e) \leq \sum_{e \in p_i^* \cap p_i} c_e(f_e) + \sum_{e \in p_i^* \setminus p_i} c_e(f_e + 1) \leq \sum_{e \in p_i^*} c_e(f_e + 1).$$

The first inequality is because f is a NE, therefore any deviation (in particular deviation to f*) results in a cost that's equal or greater. The second inequality is due to the fact that every cost function is affine and therefore monotone.

Stage 2:
By summing the inequalities from stage 1 over all players, we get:

$$c(f) = \sum_{i=1}^{k} \sum_{e \in p_i} c_e(f_e) \le \sum_{i=1}^{k} \sum_{e \in p_i^*} c_e(f_e + 1) = \sum_{e \in E} f_e^* c_e(f_e + 1) = \sum_{e \in E} a_e f_e^* (f_e + 1) + b_e f_e^*.$$

Stage 3:

Lemma 15 *(without proof) For every two integers $y, z \ge 0$,*

$$y(z+1) \le \frac{5}{3} y^2 + \frac{1}{3} z^2.$$

Substituting $y = f_e^*, z = f_e$, we get:

$$c(f) \le \sum_{e \in E} (a_e(\frac{5}{3}(f_e^*)^2 + \frac{1}{3} f_e^2) + b_e f_e^*) \le \frac{5}{3} \sum_{e \in E} f_e^*(a_e f_e^* + b_e) + \frac{1}{3} \sum_{e \in E} f_e(a_e f_e + b_e) \le \frac{5}{3} c(f^*) + \frac{1}{3} c(f).$$

So we get:

$$c(f) \le \frac{5}{2} c(f^*).$$

And that concludes the proof. ■