Algorithmic Game Theory April 6, 2016

Lecture 6

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1 Auctions

1.1 Single Parameter Environment

Our settings for the discussion of auctions is the following:

Given n players such that player i has (v_i, b_i) where v_i is the value for an item under auction and b_i is the declared value, i.e. b_i is the strategy of player i. Each players objective is to maximize the utility value

$$u_i(b_i) = x_i(b_i)v_i(b_i) - p_i(b_i)$$

The declared value (b_i) is known to all while the actual value (v_i) is private.

Definition 1 *A Mechanism is a function $M : b \to (X(b), P(b))$ where $b = (b_1, \ldots, b_n)$ is a strategy, $X(b) = (x_1, \ldots, x_n)$ is an allocation rule and $P(b) = (p_1, \ldots, p_n)$ is a payment rule.*

Our main goal is to design a mechanism with the following properties:

1. **DISC** - **D**ominant **S**trategy **I**ncentive **C**ompatible.
2. **Economical Efficiency** - Social welfare is maximized for rational players.
3. **Computational Efficiency** - The auction allocation rule can be computed in polynomial time.

Definition 2

- *An allocation rule x is **Implementable** if a payment rule p exists such that the mechanism $M(x, p)$ is DSIC.*
- *An allocation rule x is **Monotone** if for every player i and bid vector b_i, the allocation $x_i(z, b_i)$ is non-decreasing in z.*

Theorem 3 *(Myerson)*
For every Single Parameter Environment:

1. *An allocation rule x is* ***implementable*** $\iff$ *x is* ***monotone****.*
2. *If x is* ***monotone****, there is a unique payment rule p such that $M(x, p)$ is DISC (under the assumption $p_i(0, b_{-i}) = 0$, i.e. if a player bids 0, the payment will be 0).*

Proof: Let i be a player with parameters (v_i, b_i). Throughout the proof, we ease notations by using:

- $x(z) = x_i(z, b_{-i})$
- $p(z) = p_i(z, b_{-i})$

1. An allocation rule x is **implementable** $\iff$ x is **monotone**
   - x is **implementable** $\Longrightarrow$ x is **monotone**
     Assume x is *implementable*, i.e. the mechanism is DISC.
     Let z, y be a values such that $0 \le z < y$.
     (a)
     $$\underbrace{z \cdot x(z) - p(z)}_{v=z, b=z} \ge \underbrace{z \cdot x(y) - p(y)}_{v=z, b=y}$$
     (b)
     $$\underbrace{y \cdot x(y) - p(y)}_{v=y, b=y} \ge \underbrace{y \cdot x(z) - p(z)}_{v=y, b=z}$$
     The inequalities follow from the assumption that the mechanism is DISC.
     From inequality (a) we have
     $$p(y) - p(z) \ge z\,[x(y) - x(z)]$$
     From inequality (b) we have
     $$p(y) - p(z) \le y\,[x(y) - x(z)]$$
     Putting together, we obtain
     $$z\,[x(y) - x(z)] \le p(y) - p(z) \le y\,[x(y) - x(z)]$$
     In particular
     $$z\,[x(y) - x(z)] \le y\,[x(y) - x(z)]$$
     Since we assume $z < y$, is must hold that $x(y) - x(z) \ge 0$, i.e. $x(y) \ge x(z)$. Thus x is *monotone.*

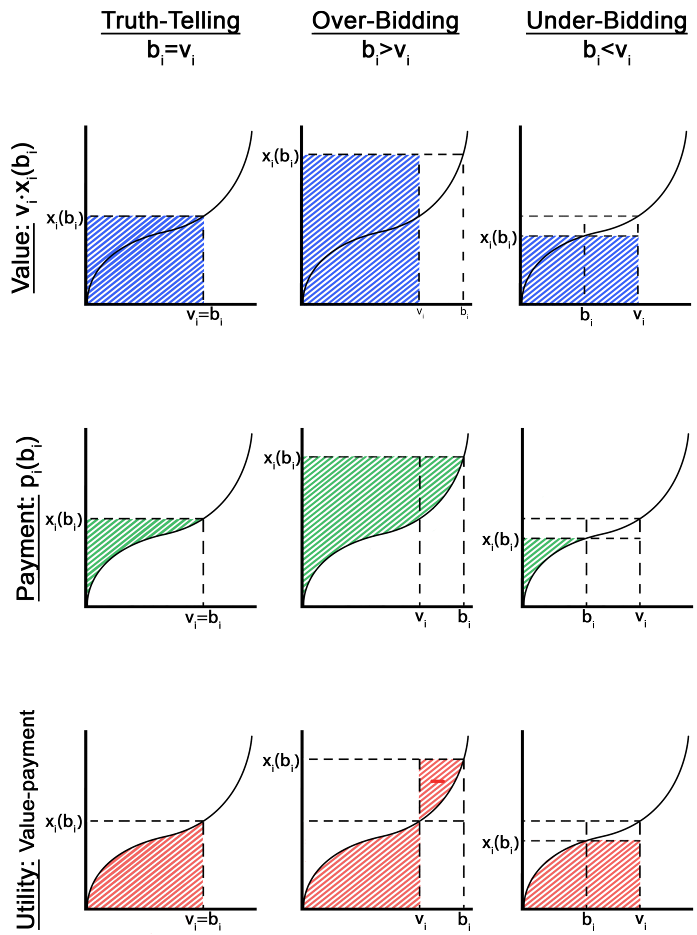

Figure 1: Strategy results

- x is **implementable** $\Longleftarrow$ x is **monotone**
  Proof by drawings (see Figure 1): It is clear from the graph that setting $b_i = v_i$ is the dominant strategy as it achieves the highest utility value.

  Note that the utility in the over-bidding case is the area under the graph **minus** the area above.

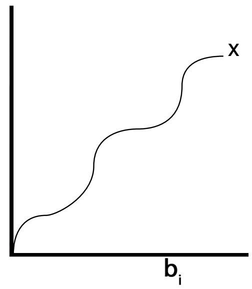

Figure 2: Graph of $x_i(z, b_{-i})$

2. Assume x is monotone. Let z, y be values such that $0 \leq z < y$. Using the analysis above, we have

$$z\,[x(y) - x(z)] \leq p(y) - p(z) \leq y\,[x(y) - x(z)]$$

Next, we divide the inequality by $(y - z)$ (note that $y > z$ so we can do such a division)

$$x\frac{[x(y) - x(z)]}{y - z} \leq \frac{p(y) - p(z)}{y - z} \leq y\frac{[x(y) - x(z)]}{y - z}$$

As $y \to z^+$ we obtain

$$\lim_{y \to z^+} z \underbrace{\frac{x(y) - x(z)}{y - z}}_{x'(z)} \leq \lim_{y \to z^+} \underbrace{\frac{p(y) - p(z)}{y - z}}_{p'(z)} \leq \lim_{y \to z^+} y \underbrace{\frac{x(y) - x(z)}{y - z}}_{x'(z)}$$

Finally, from the sandwich rule, we have

$$\forall z \quad p'(z) = z \cdot x'(z)$$

Notice that we have assumed $x \in C^1$ for the purpose of the proof.

Assume $p(0) = 0$. Recall that $x(z) \equiv x_i(z, b_{-i})$. By fixing i, b_{-i}, since x is *monotone* we have (See also Figure 2)

$$p_i\left(b_i, b_{-i}\right) = \int_0^{b_i} p'(z)dz = \int_0^{b_i} z \cdot \frac{d}{dz} x_i(z, b_{-i})dz$$

Using integration by parts we obtain

$$p_i\left(b_i, b_{-i}\right) = b_i \cdot x_i(b_i) - \int_0^{b_i} x_i(z)dz$$

So we have reached a closed form for the payment thus proving part 2 of the theorem.

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Examples

- Given the "step" function in Figure 3:
  Let $z_1, \dots, z_l$ be the "jump" points in the function $x_i(z, b_{-i})$ in the segment $[0, b_i]$. More formally,
  $$h(z_j) = \lim_{\epsilon \to 0} x_i(z_j + \epsilon, b_{-i}) - x_i(z_j - \epsilon, b_{-i})$$
  By *Myerson's* theorem the payment is:
  $$p_i(b_i, b_{-i}) = \sum_{j=1}^{l} z_j \cdot h(z_j)$$

Figure 3: Allocation "step" function

- Recall the problem of allocating banners. Assume there are k slots for k banners. The probability that a banner in slot j is clicked is α_j. Assume $\alpha_1, \dots, \alpha_k$ are in descending order.

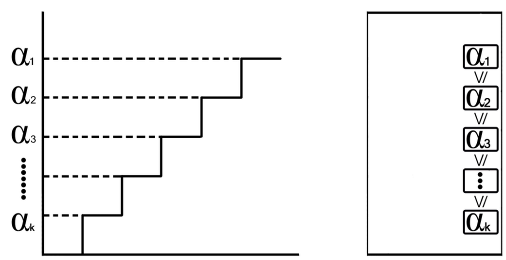

Figure 4: Banner Allocation

The allocation function is a *monotone* "step" function as described in Figure 4. The payment for player i is
$$p_i(b) = \sum_{j=i}^{k} b_{j+1} \cdot (\alpha_j - \alpha_{j+1})$$
And the payment per click is
$$p_i(b) = \sum_{j=i}^{k} b_{j+1} \cdot \frac{(\alpha_j - \alpha_{j+1})}{\alpha_i}$$

1.2 Multi-Parameter Environment

In *Multi-Parameter Environments* there are n players. A finite set of possible outcomes Ω. Each player i has a private value $v_i(\omega) \quad \forall \omega \in \Omega$.

Remark In a *Single Item Auction* $\Omega = \{\omega_0, \dots, \omega_n\}$. Where ω_i is the result of allocating the item to player i, and ω_0 is the result of not allocating the item at all.

Our goal is to build a truthful mechanism that maximizes the social welfare

$$SW(\omega) = \sum_{i=1}^{n} v_i(\omega)$$

Theorem 4 *(Vickrey-Clarke-Grove - VCG)*
There is a DISC mechanism that maximizes the Social Welfare.

Proof:
In order to prove the theorem, we present the *VCG Mechanism* and show it is DISC.

- Allocation Rule: $\omega^* = argmax_{\omega \in \Omega} \sum_{i=1}^{n} b_i(\omega)$
- Payment Rule: $p_i(b) = max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega) - \sum_{j \neq i} b_j(\omega^*)$

Note that the *Allocation Rule* is defined to maximize the *social welfare*. The *Payment Rule* for player i is the difference between the maximum *social welfare* obtained by not considering player i to the maximum *social welfare* of all players.

Next, we show that the *VCG Mechanism* is DISC.

Given player i, and payments for the rest of the players b_{-i},

$$u_i(b) = v_i(\omega^*) - p_i(b) = v_i(\omega^*) - max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega) + \sum_{j \neq i} b_j(\omega^*)$$

Since the objective of player i is to maximize $u_i(b)$ and the expression $max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega)$ is not affected by player i. Player i's objective is therefore

$$\max_{b_i} u_i(b_i) = \max_{b_i} \; v_i(\omega^*(b_i)) + \sum_{j \neq i} b_j(\omega^*(b_i))$$

One can notice that this is exactly the *Social Welfare* which is maximized by the mechanism. This means that in order for player i to maximize $u_i(b_i)$, the dominant strategy is exactly $b_i = v_i$ - the mechanism is DISC.

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Examples

1. ***Single Item Auction***

   Using *VCG*'s mechanism will result with the 2^{nd} *Price Auction* as the *Allocation Rule* is to give the item to the highest bidder. The *Payment Rule* is defined as $p_i(b) = max_\omega \in \Omega \sum_{i \neq j} b_j(\omega) - \sum_{i \neq j} b_j(\omega^*)$ which means the buyer pays the bid of the second highest bidder while all other players pay 0.

2. ***K Identical Items***

   Assume $n > k$. There are k identical products. Each player wants 1 product. W.L.O.G $v_1 > v_2 > \cdots > v_n$. Players $1, \ldots, k$ are allocated with an item by the *VCG* mechanism. The cost $p_i(b) = \sum_{j=1, j \neq i}^{k+1} b_j - \sum_{j=1, j \neq i}^{k} b_j = b_{k+1}$. All other players pay 0.

3. ***Public Goods***

   Alice, Bob, Carol and *Dorothy* are citizens of the infamous city *Computer Sciencity.* The mayor is facing the decision whether to build a bridge for the citizens welfare. Building the bridge costs 1000. The benefit of the citizens: $V_A = 300$, $V_B = 400$, $V_C = 300$, $V_D = 200$
   Using *VCG*'s mechanism, the *Allocation Rule* is

   $$SW = \sum_{i \in \{A,B,C,D\}} V_i - \underbrace{1000}_{Bridge\ Cost}$$

   Which means the bridge is built $\iff \sum_{i \in \{A,B,C,D\}} V_i - 1000 > 0$
   The *Payment Rule* is:

   - $P_A = P_C = 0 - (900 - 1000) = 100$
   - $P_B = 0 - (800 - 1000) = 200$
   - $P_D = 0 - (1000 - 1000) = 0$

4. ***Procurement Auction***

   There are n suppliers for an item. Supplier i pays for the item v_i. *Social Welfare* is maximized by buying from the supplier with the lowest price.

   For example $V_A = -17$, $V_B = -20$, $V_C = -30$ are the suppliers cost of the item. *Social Welfare* is obtained by buying from supplier A.

   $$P_A = -20 - 0 = -20$$

   So the price of the item from supplier A is set to 20 by the *VCG* mechanism.

2 Combinatorial Auctions

There are n rational players.
There are m items in group M
Each player i has a validation function

$$V_i : 2^{[m]} \to \mathbb{R}^{\geq 0}$$

Assume

1. ***Normalization***
$$V_i(\emptyset) = 0$$

2. ***Monotonicity, Free Disposal***

$$\forall S \subseteq T \subseteq M \qquad V_i(S) \leq V_i(T)$$

The goal is to maximize *Social Welfare.*

Definition 5 *Allocation Rule is a collection of sets $X = (X_1, \ldots, X_n)$ such that $\forall i\ X_i \subseteq M$.*

$$\forall i \neq j X_i \cap X_j = \emptyset \qquad \bigcup_{i=1}^{n} X_i = M$$