Algorithmic Game Theory	May 4, 2016

Lecture 9

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1 VCG mechanism

We defined VCG mechanism in the following way:

- Allocation rule: $\omega^* = argmax_{\omega \in \Omega} \sum_{i=1}^{n} b_i(\omega)$
- Payment rule: $p_i(b_i) = max_{\omega \in \Omega} \sum_{i \neq j} b_i(\omega) - \sum_{i \neq j} b_i(\omega^*)$

Remark VCG is a family of mechanisms. The mechanism we defined is called "Clarke-Pivot" mechanism, and it is a private case of the VCG mechanisms.

The VCG mechanism is a DISC mechanism that maximizes the social welfare (SW). However, finding an allocation that maximizes $argmax_{\omega \in \Omega} \sum_{i=1}^{n} b_i(\omega)$ might be a computationally hard problem (we saw that for single-minded bidders maximizing the social welfare is a NP-Hard problem). Last lecture we saw the maximum in range (MIR) methodology, which uses VCGs mechanism on a restricted range of allocations. This method improves the computational complexity of the allocation rule, and reaches a 2-approximation to the maximal social welfare. We saw that this is the best approximation we can get by using the MIR method.

2 Non-truthful Mechanisms

In the real world the mechanisms are not necessarily truthful (for example, we saw that the generalized second price auction used by Google for sponsored search is not a truthful mechanism). We will analyze the price of anarchy caused by non-truthful bidding, i.e we will find the largest relation between the SW in some Nash equilibrium and the optimal SW.

2.1 Simultaneous second price auction (S2A)

Lets consider the following auction:

- A set M of m goods.
- A set N of n players.
- Each player i has a valuation function $V_i = 2^M \to R$

VCG is an inefficient mechanism for this auction. We will suggest the following simple mechanism:

Definition 1 *Simultaneous second price auction (S2A):*

- *Every player i submits a bid for each good j separately ($b_{i,j}$). The mechanism gets a vector of bids from each player i - ($b_{i,1}, b_{i,2}...b_{i,m}$).*
- *The mechanism applies the second price allocation and payment rules for each separately.*

Note that if the valuation function of the players are additive, the SIA mechanism is exactly the VCG mechanism, and it is therefor a DISC mechanism that maximizes the SW. Otherwise, the mechanism is not necessarily truthful. Lets consider the following case:

Example 1

- *There is a single good p and two players (player1 and player2).*
- *We will denote V_i to be player i's valuation function. $V_1(p) = 1, V_2(p) = \epsilon, 0 < \epsilon < 1$.*

Lets view the case where player1 bids 0, player2 bids 1. This is a Nash equilibrium:

1. *Player 2 wins and pays 0. In this case the utility of player 2 is maximized, and he can't gain anything by changing his bid.*
2. *Player1 loses and his utility is 0. Player 1 can win only if he would change his bid to be higher than 1. However, in this case the his utility would be lower than 0, and he won't gain anything from changing his bid.*

The SW in this case is ϵ. In the optimal case, player1 bids 1 and player2 bids 0, and the SW is 1. We get that the price of anarchy is : $PoA = SW_{OPT}/SW_{NE} = 1/\epsilon$. Since ϵ can be as small as we want, we get that the price of anarchy is as big as we want it to be.

The above example seems unrealistic - player1's bid is much higher than his valuation for p, which is nearly 0. We will make the following assumptions:

1. No over bidding (NOB): Lets denote $S_i(b)$ to be the set of goods that player i receives with the bidding vector b. A vector b holds the NOB property if for every player i: $\sum_{j \in S_i(b)} b_{i,j} \leq V_i(S_i(b))$.

2. Sub-modular valuations: A valuation function V_i is sub-modular if for every $S \subseteq T \subseteq M$ and for every j : $V_i(T \cup \{j\}) - V_i(T) \leq V_i(S \cup \{j\}) - V_i(S)$ (player i gains more from receiving j in addition to the set S, than it does from receiving j in addition to T).

Note that the unit demand function is sub modular.

Example 2 *Lets consider the following auction:*

- *There are two goods , X and Y, and two players (player1 and player2).*
- *The valuation functions are defined as follows:*
  - $V_1(X) = 2, V_1(Y) = 1$
  - $V_2(Y) = 2, V_2(X) = 1$

If X is allocated to player1 and Y is allocated to player2 we get $SW = 4$, this is the optimal SW.

Lets view the scenario where player 1 bids $b_1(X) = 0, b_1(Y) = 1$, and player2 bids $b_2(X) = 1, b_2(Y) = 0$ (note that this bidding vector holds the NOB property). We will see that this case is a Nash equilibrium:

- *Player1 pays 0 for Y (since player2 bid is 0), and his utility is 1*
- *To win X , player1 needs to change his bid for X to be higher than 1, and is payment for X would be 1. $V_1(\{X, Y\}) = max(V_1(X), V_1(Y)) = 2$, his utility would be 1, and he won't gain anything from this change.*
- *Since player1 gets Y for free, he won't gain anything from changing his bid for Y .*
- *The same analysis applies for player2.*

The SW in this case is 2, and so we get that $PoA = SW_{OPT}/SW_{NE} = 4/2 = 2$.

We will see that the PoA in example2 is the worst case for any scenario which holds the NOB and sub-modular properties.

Theorem 2 *Let v_i be a sub-modular valuation function of the i^{th} player, and b an equilibrium of the SIA under the NOB assumption. Than $SW(b) = \sum_i v_i(s_i(b)) \geq \frac{SW^*(b)}{2}$ where $SW^*(b)$ is the optimal SW.*

We will first prove the theorem for unit demand valuation as a warm up:
Proof: Let b be an equilibrium biding profile (under NOB assumption), and let S^* be an

optimal allocation: $OPT = \sum_{i=1}^{n} v_i(s_i^*)$. Let $j^*(i)$ be the product given to the i^{th} player in OPT (Without loss of generality we can assume each player receives one product).
Consider the following strategy for the i^th player:

$$b_{ij}^* = \begin{cases} v_{ij} & j = j^*(i) \\ 0 & O.W. \end{cases}$$

Since b is equilibrium, no player can increase his profit by changing strategy. Therefore:

$$v_i(s_i(b)) - \sum_{j \in s_i(b)} p_j(b) \geq v_i(s_i(b_i^*, b_{-i})) - \sum_{j \in s_i(b_i^*, b_{-i})} p_j(b_i^*, b_{-i})$$

And since p_j is non-negative, we can write:

$$v_i(s_i(b)) \geq v_i(s_i(b_i^*, b_{-i})) - \sum_{j \in s_i(b_i^*, b_{-i})} p_j(b_i^*, b_{-i})$$

Notice that if $v_{ij} \geq \max_{k \neq i} b_{kj}$ than the i^{th} player gets the j^{th} product in the (b_i^*, b_{-i}) profile in price $\max_{k \neq i} b_{kj}$. Otherwise, $v_{ij} - \max_{k \neq i} b_{kj} < 0$ and the i^{th} player is not getting the product and not paying anything. Therefore, in any case the contribute of the $j^*(i)$ product to $v_i(s_i(b_i^*, b_{-i})) - \sum_{j \in s_i(b_i^*, b_{-i})} p_j(b_i^*, b_{-i})$ is at least $v_{ij} - \max_{k \neq i} b_{kj} \geq v_{ij} - \max_{k} b_{kj}$.
Overall we get:

$$v_i(s_i(b)) \geq v_{ij^*(i)} - \max_{k} b_{kj}$$

And the sum over all players:

$$(1) \sum_{i=1}^{n} v_i(s_i(b)) \geq \sum_{i=1}^{n} v_{ij^*(i)} - \max_{k} b_{kj^*(i)}$$

Notice the following:

$$\sum_{i=1}^{n} \max_{k} b_{kj^*(i)} \leq \sum_{i=1}^{n} \sum_{j \in s_i(b)} \max_{k} b_{kj}$$

And since we know that the i^{th} player receives the j^{th} product, and under the NOB assumption we get:

$$= \sum_{i=1}^{n} \sum_{j \in s_i(b)} b_{ij} \leq \sum_{i=1}^{n} v_i(s_i(b))$$

We can now replace it back in equation (1):

$$\underbrace{\sum_{i=1}^{n} v_i(s_i(b))}_{SW@NE} \geq \underbrace{v_{ij^*(i)}}_{OPT} - \underbrace{\sum_{i=1}^{n} v_i(s_i(b))}_{SW@NE}$$

And therefore $SW@NE \geq \frac{OPT}{2}$
■

Lemma 3 *Let u_i be a submodular function, such there are additive functions - $a_i^1, ..a_i^l, a_i^r$, such that, for each set S:*

$$v_i(S) = max_{l=1}^r \{\sum_{j \in S} a_{ij}^l\}$$

where

$$a_{ij}^l = (a_{i1}^l, ..., a_{im}^l)$$

Proof: Lets view the $m!$ additive functions of every possible permutation. For every permutation:

$$\pi : a_{ij}^\pi = v_i(S_j^\pi \cup \{j\}) - v_i(S_j^\pi)$$

We will take a look at the set S. We will fix a permutation π (meaning, some additive function), Then we'll sort the members of S by their order in permutation π:

$$a_i^\pi(S) = \sum_{j \in S} a_{ij}^\pi = \sum_{j \in S}(v_i(S_j^\pi \cup \{j\}) - v_i(S_j^\pi)) \leq \sum_{j \in S}(v_i(1, ..., j-1)) = v_i(S) - v_i(\emptyset) = v_i(S) = v_i(S)$$

Every permutation that it's prefix contains exactly the memebers of the set S, will give us the equailty, therefore such π exists. ■

Now We'll proof the theorem $PoA \leq 2$ for submodular functions. Let b be a nash equilibrium with NOB. Let S^* be an optimal allocation, and let S_i^* be the alloction of player i in OPT. By the lemma, for every player i^{th}, there exists an additive function a_i^* such that $v_i(S_i^*) = \sum_{j \in S_i^*}(a_{ij}^*)$.
We'll use the hypothetical deviation:

$$b_{ij}^* = \begin{cases} a_{ij}^* & j \in S_i^* \\ 0 & O.W. \end{cases}$$

from the equilibrium we get that:

$$v_i(s_i(b)) - \sum_{j \in s_i(b)} p_j(b) \geq v_i(s_i(b_i^*, b_{-i})) - \sum_{j \in s_i(b_i^*, b_{-i})} p_j(b_i^*, b_{-i})$$

And since p_j is non-negative, we can write:

$$v_i(s_i(b)) \geq v_i(s_i(b_i^*, b_{-i})) - \sum_{j \in s_i(b_i^*, b_{-i})} p_j(b_i^*, b_{-i})$$

For every $j \in S_i^*$, if $a_{ij} \geq \max_{k \neq i} b_{kj}$, the i^{th} gets the j^th product and pays $\max_{k \neq i} b_{kj}$. Otherwise, $a_{ij} < \max_{k \neq i} b_{kj}$ and then the i^{th} player gets nothing.
Each product $j \in S_i^*$ contibutes at least $a_{ij}^* - \max_{k \neq i} b_{kj} \geq a_{ij}^* - \max_{k=1}^n b_{kj}$

Overall we get:

$$v_i(s_i(b_i^*, b_{-i})) - \sum_{j \in s_i(b_i^*, b_{-i})} p_j(b_i^*, b_{-i}) \geq \sum_{j \in s_i^*} (a_{ij}^* - \max_{k=1}^{n} b_{kj})$$

Overall we get that for every i:

$$v_i(s_i(b)) \geq \sum_{j \in s_i^* a_{ij}^*(i)} - \max_{k=1}^{n} b_{kj}$$

And the sum over all players:

$$\sum_{i=1}^{n} v_i(s_i(b)) \geq \sum_{i=1}^{n} a_{ij^*(i)} - \max_{k} b_{kj^(i)}$$

According To The Lemma

$$v_i(S) \geq \sum_{j \in S} a_{ij}^*$$

so we get:

$$(1) \underbrace{\sum_{i=1}^{n} v_i(s_i(b))}_{SW@NE} \geq \underbrace{\sum_{i=1}^{n} v_i(s_i^*)}_{OPT} - \underbrace{\sum_{i=1}^{n} \sum_{j \in S_i^*} \max_{k=1}^{n} b_{kj}}_{\leq SW@NE}$$

Since each product is given to one player:

$$\sum_{i=1}^{n} \sum_{j \in S_i^*} \max_{k=1}^{n} b_{kj} \leq \sum_{i=1}^{n} \sum_{j \in S_i(b)} \max_{k=1}^{n} b_{kj}$$

And since we know that the i^{th} player receives the j^{th} product, and under the NOB assumption we get:

$$= \sum_{i=1}^{n} \sum_{j \in s_i(b)} b_{ij} \leq \sum_{i=1}^{n} v_i(s_i(b))$$

We can now replace it back in equation (1):

$$\underbrace{\sum_{i=1}^{n} v_i(s_i(b))}_{SW@NE} \geq \underbrace{v_{ij^*(i)}}_{OPT} - \underbrace{\sum_{i=1}^{n} v_i(s_i(b))}_{SW@NE}$$

And therefore $SW@NE \geq \frac{OPT}{2}$